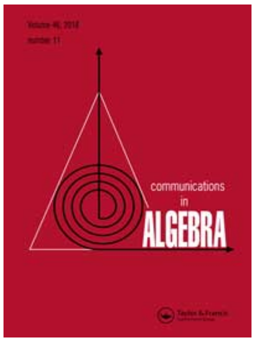

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

Lifting of automorphisms of factor modules

Adel Nailevich Abyzov & Cong Quynh Truong

COMMUNICATIONS IN ALGEBRA®
2018, VOL. 46, NO. 11, 5073–5082
https://doi.org/10.1080/00927872.2018.1461884

Lifting of automorphisms of factor modules

Adel Nailevich Abyzov[a] and Cong Quynh Truong[b,c]

[a]Department of Algebra and Mathematical Logic, Kazan (Volga Region) Federal University, Kazan, Russia; [b]Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam; [c]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

ABSTRACT

This paper introduces the notion of dual of automorphism extendable modules. A module M is called automorphism-extendable if for every submodule N of M, every automorphism of N can be extended to an endomorphism of M. We call a module M a dual automorphism-extendable module if whenever K is a submodule of M, then every automorphism $\nu : M/K \to M/K$ lifts to an endomorphism θ of M. In this paper we give various examples of dual automorphism-extendable modules and study their properties. In particular, we prove that every dual automorphism-extendable module is a D3-module. It is shown that over a right artinian ring R, an R-module $M = \oplus_I M_i$ with hollow modules M_i is dual automorphism-extendable if and only if M is quasi-projective.

ARTICLE HISTORY
Received 7 November 2016
Revised 26 March 2018
Communicated by T. Albu

KEYWORDS
Artinian ring; dual automorphism-extendable module; dual automorphism-invariant module; perfect ring; quasi-projective module

2000 MATHEMATICS SUBJECT CLASSIFICATION
16D40; 16D80; 16N80

1. Introduction and preliminaries

All our rings will be associative rings with identity and modules will be unitary right modules. A module M is called an automorphism-invariant module if every isomorphism between two essential submodules of M extends to an automorphism of M (see [13]). The class of automorphism-invariant modules was studied in several papers (see [5, 9, 12, 16, 21]). A right R-module M is called an automorphism-extendable module if every automorphism of any submodule can be extended to an endomorphism of M. Equivalently, if for every essential submodule N of M, each automorphism of N can be extended to an endomorphism of M. The characterization of automorphism-extendable modules was studied in the papers [18–21]. In [21, Theorem 1], it is proved that a semi-artinian module M is automorphism-invariant if and only if M is automorphism-extendable.

A submodule A of a module M is called small in M (denoted by $A \ll M$) if $A + B \neq M$ for every proper submodule B of M. A module M is called a hollow module if all proper submodules of M are small in M. The Jacobson radical of a module M is the sum of all small submodules of M and is denoted by $J(M)$. A module M is called N-projective (projective relative to N) if for every submodule A of N, every homomorphism $\alpha : M \to N/A$ can be lifted to a homomorphism $\beta : M \to N$. If M is M-projective then M is said to be a quasi-projective module. And if, M is N-projective for all right R-modules N then M is called a projective module. An epimorphism $\varphi : P \to M$ with P a projective module and $\mathrm{Ker}(\varphi) \ll P$ is called a projective cover of M. We refer to [2, 4, 6, 7, 14, 15] for any undefined notion used in the text.

A module M is said to be a dual automorphism-invariant module if whenever K_1 and K_2 are small submodules of M, then every epimorphism $\varphi : M/K_1 \to M/K_2$ with small kernel lifts to an endomorphism of M (see [17]). Some properties of dual automorphism-invariant modules were studied in the papers [8, 11, 17].

CONTACT Cong Quynh Truong truongcongquynh@tdt.edu.vn Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam; Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.

In this paper, we introduce the notion of dual automorphism-extendable modules and study some properties of them.

Definition. A right R-module M is called a dual automorphism-extendable module if whenever K is a submodule of M, then every automorphism ν of M/K lifts to an endomorphism θ of M.

$$\begin{tikzcd} M \arrow[r, "\theta"] \arrow[d] & M \arrow[d] \\ M/K \arrow[r, "\eta"] & M/K \end{tikzcd}$$

Example 1. (i) All quasi-projective modules are dual automorphism-extendable. In particular, all semisimple modules are dual automorphism-extendable.

(ii) For every prime number p, the Prüfer group $\mathbb{Z}_{p^\infty}$ is dual automorphism-extendable as a $\mathbb{Z}$-module. Note that $\mathbb{Z}_{p^\infty}$ is not a dual automorphism-invariant module by [17, Theorem 20]. Furthermore, $\mathbb{Q}/\mathbb{Z}$ is dual automorphism-extendable but not dual automorphism-invariant.

(iii) Let $Q = \prod_{i=1}^{\infty} F_i$ where $F_i = \mathbb{Z}_2$ and T the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then T is a commutative, non-semisimple V-ring. It follows that every finitely generated module is dual automorphism-invariant by [17, Theorem 5]. But by Theorem 10 and [1, Proposition 30], there is a finitely generated R-module that is not dual automorphism-extendable.

2. Some properties of dual automorphism-extendable modules

We first show that the class of dual automorphism-extendable modules is closed under direct summands.

Lemma 2. *Every direct summand of a dual automorphism-extendable module is dual automorphism-extendable.*

Proof. Let $M = N \oplus H$ be a dual automorphism-extendable module. We will show that N is a dual automorphism-extendable module. Let K be an arbitrary submodule of N and θ an automorphism of N/K. There exists an isomorphism $\phi : N/K \to M/(K+H)$ via $\phi(\bar{n}) = \bar{n}$ for all $\bar{n} \in N/K$. So, $\theta' = \phi \circ \theta \circ \phi^{-1}$ is an automorphism of $M/(K+H)$. As M is a dual automorphism-extendable module, there exists an endomorprhism α' of M such that $p_2 \circ \alpha' = \theta' \circ p_2$ with $p_2 : M \to M/(K+N')$ the natural projection. Call $p_1 : N \to N/K$ the natural projection, $\iota : N \to M$ the inclusion map and $\pi : M \to N$ the canonical projection. It follows that the following squares are commutative:

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[d, "p_1"'] & M \arrow[r, "\alpha'"] \arrow[d, "p_2"'] & M \arrow[r, "\pi"] \arrow[d, "p_2"'] & N \arrow[d, "p_1"'] \\ N/K \arrow[r, "\phi"] & M/(K+N') \arrow[r, "\theta'"] & M/(K+N') \arrow[r, "\phi^{-1}"] & N/K \end{tikzcd}$$

Call $\alpha = \pi \circ \alpha' \circ \iota : N \to N$. Then $\theta \circ p_1 = p_1 \circ \alpha$ competing the proof. □

We can check easily the following result:

Lemma 3. *Let M be a dual automorphism-extendable module. If X is a submodule of M and invariant under all endomorphisms of M, then M/X is also a dual automorphism-extendable module.*

A module M is called *supplemented* if for every submodule N of M there exists a submodule L of M with $N + L = M$ and $N \cap L \ll L$.

Lemma 4. *A supplemented right R-module M is dual automorphism-extendable if and only if for every small submodule X in M, every automorphism of M/X can be lifted to an (surjective) endomorphism of M.*

Proof. We only need to prove the sufficiency. Let X be an arbitrary submodule of M. By our assumption, there is a submodule Y of M such that $M = X + Y$ and $X \cap Y \ll M$. Then

$$M/X \simeq Y/(X \cap Y) \text{ and } M/(X \cap Y) = X/(X \cap Y) \oplus Y/(X \cap Y).$$

We will show that every automorphism of M/X is lifted to an endomorphism of M. By the natural isomorphism between M/X and $Y/(X \cap Y)$, it suffices to show that every automorphism α of $Y/(X \cap Y)$, there exists an endomorphism ψ of M such that $\alpha \circ \pi = \pi \circ \psi$, where the epimorphism $\pi : M \to Y/(X \cap Y)$ is defined by $\pi(x + y) = y + (X \cap Y)$ for all $x \in X, y \in Y$. In fact, let α be an arbitrary automorphism of $Y/(X \cap Y)$. Call the isomorphism $\phi = 1_{X/(X \cap Y)} \oplus \alpha : M/(X \cap Y) \to M/(X \cap Y)$. By our assumption, there exists an endomorphism $\psi : M \to M$ such that $\phi \circ p = p \circ \psi$ with $p : M \to M/(X \cap Y)$ the natural projection. Thus $\alpha \circ \pi = \pi \circ \psi$. □

Corollary 5. *Assume that M is a supplemented module such that $1 - \alpha$ is an automorphism of M whenever α is an endomorphism of M with small kernel. Then M is dual automorphism-extendable if and only if for every small submodule X in M, then every automorphism of M/X is lifted to an automorphism of M.*

Proof. Assume that M is a dual automorphism-extendable module. Let $X \ll M$ and $\alpha : M/X \to M/X$ an isomorphism. Then there exist homomorphisms $\beta_1, \beta_2 : M \to M$ such that $\alpha \circ p = p \circ \beta_1$ and $\alpha^{-1} \circ p = p \circ \beta_2$, where $p : M \to M/X$ is the natural projection. Inasmuch as $X \ll M$, then h_1, h_2 are epimorphisms with small kernels. It is easy to check that $\mathrm{Ker}(h_1 \circ h_2) \ll M$ and $\mathrm{Ker}(h_2 \circ h_1) \ll M$. Moreover, we obtain $p \circ (1 - h_1 \circ h_2) = 0$ and $p \circ (1 - h_2 \circ h_1) = 0$. Assume that either $h_1 \circ h_2 \neq 1$ or $h_2 \circ h_1 \neq 1$. By our assumption, $1 - h_1 \circ h_2$ or $1 - h_2 \circ h_1$ is an isomorphism. It follows that $p = 0$, a contradiction. Thus $h_1 \circ h_2 = 1$ and $h_2 \circ h_1 = 1$. □

Proposition 6. *If $M = X \oplus Y$ is a dual automorphism-extendable module, then X and Y are relatively projective.*

Proof. Let A be a submodule of Y and $f : X \to Y/A$, an R-homomorphism. We will prove that f can be lifted to Y. Let $\pi : M \to M/A$ be the natural projection, $\overline{M} = M/A$ and $\overline{S} = (S + A)/A$ for any $S \leq Y$. Then $\overline{M} = \overline{X} \oplus \overline{Y}$. Let $\overline{f} : \overline{X} \to \overline{Y}$ be the homomorphism induced by f in an obvious way. Now, $\overline{M} = (\iota + \overline{f})(\overline{X}) \oplus \overline{Y}$, where $\iota : \overline{X} \to \overline{M}$ is the inclusion map. Then the map $g : \overline{M} \to \overline{M}$ with $g = (\iota + \overline{f}) \oplus 1_{\overline{Y}}$ is an automorphism of $\overline{M}$, which lifts to an endomorphism h of M, so that, in particular, $h(x) + A = (x + A) + f(x)$ $(x \in X)$, whence $h(x) - x + A \in \overline{Y}$, and thus $h(x) - x \in Y$. So if $h(x) = x' + y'$ $(x' \in X', y' \in Y)$, we must have $x = x'$, and hence $f(x) = y' + A = (\pi \circ \eta_Y \circ h)(x)$, where $\eta_Y : X \oplus Y \to Y$ is the canonical projection. Therefore $\pi \circ \eta_Y \circ h$ is the desired map. It shows that X is Y-projective. The X-projectivity of Y is proved similarly. □

Remark 7. We do not know if the direct sum of two relatively projective dual automorphism-extendable modules is again dual automorphism-extendable.

Example 8. (1) The $\mathbb{Z}$-modules $\mathbb{Z}_2$ and $\mathbb{Z}$ are dual automorphism-extendable modules but $\mathbb{Z}_2 \oplus \mathbb{Z}$ is not dual automorphism-extendable.

(2) The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ is not dual automorphism-extendable, while $\mathbb{Z}_{p^\infty}$ is dual automorphism-extendable.

A module M is called a *D3-module* if whenever M_1 and M_2 are direct summands of M and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M. This class of D3-modules was thoroughly studied in [**?**]. In [17], the authors proved that every dual automorphism-invariant module M is a D3-module if M is a supplemented module. The $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_3$ is a dual automorphism-invariant module that is not a D3-module. In the next result we show that every dual automorphism-extendable module is a D3-module.

Lemma 9 ([4, 4.12]). *The following are equivalent for $M = M_1 \oplus M_2$.*
(1) *M_1 is M_2-projective.*
(2) *For every submodule N of M such that $M = N + M_2$, there exists a submodule N_0 of N such that $M = N_0 \oplus M_2$.*

Theorem 10. *Every dual automorphism-extendable R-module is a D3-module.*

Proof. Let M be a dual automorphism-extendable module. Let A and B be direct summands of M such that $A + B = M$. We will show $A \cap B$ is a direct summand of M. As B is a direct summand of M, we can write $M = B' \oplus B$ for some submodule B' of M. By our assumption, M is a dual automorphism-extendable module and hence B' is B-projective by Proposition 6. By Lemma 9, there exists a submodule N of A such that M has a decomposition $M = N \oplus B$. It follows that $A = N \oplus (A \cap B)$. But A is a direct summand of M, $A \cap B$ is also a direct summand of M. Thus, M is a D3-module. □

3. On dual automorphism-extendable abelian groups

We next consider dual automorphism-extendable abelian groups. In particular, we study torsionfree and torsion dual automorphism-extendable abelian groups.

Recall that an abelian group M is said to be a *divisible* group if for every positive integer t and every element $m \in M$, there exists $m' \in M$ such that $m = tm'$. An abelian group M is called a *reduced* group if M has no proper divisible subgroups.

Fact 11. (i) Let $M_{\mathbb{Z}}$ be a submodule of $\mathbb{Q}$ and containing $\mathbb{Z}$. Assume that M is a dual automorphism-extendable module. Then from the proof of Theorem 22 in [17], M is dual automorphism-invariant and $J(M) = 0$.
(ii) All abelian groups $\mathbb{Z} \oplus \mathbb{Z}_n$ for all positive integer n are dual automorphism-invariant $\mathbb{Z}$-modules. Moreover, if $\mathbb{Z}_n \neq 0$ then $\mathbb{Z} \oplus \mathbb{Z}_n$ is not dual automorphism-extendable by Proposition 6.

Proposition 12. *Every torsionfree dual automorphism-extendable abelian group is reduced.*

Proof. Assume that M is a torsionfree dual automorphism-extendable abelian group. It follows that $M = (\bigoplus_I \mathbb{Q}) \oplus K$ for some reduced group K. As $\mathbb{Q}$ is not dual automorphism-extendable by Fact 11. We deduce that $M = K$ is a reduced group. □

In case of torsion abelian groups, we have the following result:

Proposition 13. *Let M be a torsion abelian group. The following conditions are equivalent:*
(1) *M is dual automorphism-extendable.*
(2) *M has a decomposition $M = (\oplus_{p\in I}\mathbb{Z}_{p^\infty}) \oplus (\oplus_{p\in I'}(\oplus\mathbb{Z}_{p^{m_p}}))$ (m_p fixed integers) for some disjoint subsets I, I' of the set of all prime numbers.*

Proof.
(1) $\Rightarrow$ (2). Assume that M is a torsion dual automorphism-extendable abelian group. Then M is a direct sum of a divisible group and a reduced group. Note that the group $\mathbb{Q}$ is not dual

automorphism-extendable and $\mathbb{Z}_{p^\infty}$ is not quasi-projective for every prime number p. Then there exist a reduced group K and a subset I of the set of all prime numbers such that $M = (\bigoplus_{p\in I} \mathbb{Z}_{p^\infty}) \oplus K$. We claim that every p-component K_p of K is a direct sum of cyclic groups of the same order p^n. In fact, let K_p be a p-component of K and B, a basic subgroup of K_p. By definition of B, $B = \bigoplus_{x\in L} x\mathbb{Z}$ for some subset L of K_p. For every $x, y \in L$ and $x \neq y$, call p^{k_1} and p^{k_2} orders of x and y respectively. Assume that $k_1 \neq k_2$. Then $\mathbb{Z}_{p^{k_1}} \oplus \mathbb{Z}_{p^{k_2}}$ is a dual automorphism-extendable module. So, $\mathbb{Z}_{p^{k_1}}$ and $\mathbb{Z}_{p^{k_2}}$ are relatively projective, a contradiction. It shows that for every element of L has the same order p^n. We deduce that B is a direct summand of K_p by [7, Proposition 27.1]. But K_p is a reduced group and hence $K_p = B$. It follows that $K = \bigoplus_{p\in I'} (\oplus \mathbb{Z}_{p^{m_p}})$ (m_p fixed integers) for some subset I' of the set of all prime numbers. Inasmuch as $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^n}$ is not a dual automorphism-extendable module for every prime p, we have to get $I \cap I' = \emptyset$.

(2) $\Rightarrow$ (1) is obvious. □

4. Dual automorphism-extendable modules over artinian rings

We will consider the question of when the classes of dual automorphism-invariant modules and that of dual automorphism-extendable modules coincide.

Proposition 14. *Assume that M is a supplemented module. If M is a dual automorphism-invariant module, then M is dual automorphism-extendable.*

Proof. Let X be a submodule of M and $f : M/X \to M/X$ an isomorphism. By our assumption, there exists a submodule Y of M such that $X + Y = M$ and $X \cap Y \ll M$. Note that

$$\frac{M}{X\cap Y} = \frac{X}{X\cap Y} \oplus \frac{Y}{X\cap Y}.$$

Let $\pi : X/(X \cap Y) \oplus Y/(X \cap Y) \to Y/(X \cap Y)$ and $g : M/(X \cap Y) \to M/X$ be the canonical projection and the quotient map respectively. We have $\mathrm{Ker}(\pi) = \mathrm{Ker}(g)$ and obtain that there exists an isomorphism $\phi : Y/(X \cap Y) \to M/X$ such that $\phi \circ \pi = g$. Let $\varphi : M/(X \cap Y) \to M/(X \cap Y)$ be the isomorphism defined by

$$\varphi|_{X/(X\cap Y)} = id_{X/(X\cap Y)} : X/(X \cap Y) \to X/(X \cap Y)$$

$$\varphi|_{Y/(X\cap Y)} = \phi^{-1} \circ f \circ \phi : Y/(X \cap Y) \to Y/(X \cap Y).$$

It is easy to see that $\phi^{-1} \circ f\phi \circ \pi = \pi \circ \varphi$. Thus $f \circ g = g \circ \varphi$. As M is a dual automorphism-invariant module, there exists an endomorphism $\varphi' : M \to M$ such that $\varphi \circ p = p \circ \varphi'$, where $p : M \to M/X \cap Y$ is the natural projection. Note that $p_1 = g \circ p : M \to M/X$ is the natural projection. Thus $f \circ p_1 = p_1 \circ \varphi'$. □

The radical series $J^\alpha(M)$ is defined inductively for each ordinal α in the usual way, where $J(M)$ is the intersection of all maximal submodules of M,

$$J^0(M) = M$$

$$J^{\alpha+1}(M) = J(J^\alpha(M)) \text{ for every ordinal } \alpha$$

$$J^\beta(M) = \bigcap_{\alpha<\beta} J^\alpha(M) \text{ for each limit ordinal } \beta.$$

We now suppose that M is a dual automorphism-extendable module. By the definition of M, then every automorphism α of its factor with respect to small submodules can be lifted to an endomorphism $\bar{\alpha}$ of M. In general, we don't know whether $\bar{\alpha}$ is an automorphism of M or not. We will deal with this question in the next proposition, but first recall that a module M is called *hopfian* if every surjective endomorphism is an automorphism.

Proposition 15. *Assume that R is a right perfect ring. Then:*

(1) *The following conditions are equivalent for a hopfian R-module M:*
   (a) *M is a dual automorphism-invariant module.*
   (b) *M is a dual automorphism-extendable module.*

(2) *The following conditions are equivalent for an R-module M:*
   (a) *M is a dual automorphism-invariant module.*
   (b) *For any two small submodules K_1 and K_2 of M, any automorphism $\phi : M/K_1 \to M/K_2$ lifts to an automorphism of M.*
   (c) *For any small submodule K of M, any automorphism $\phi : M/K \to M/K$ lifts to an automorphism of M.*

Proof.

(1) $(a) \Rightarrow (b)$ by Proposition 14.

$(b) \Rightarrow (a)$ Suppose that $M = P/K$ is a dual automorphism-extendable module, where P is a projective module and $K \ll P$. Let $f : P \to P$ be an isomorphism. We will claim that $f(K) = K$. Let $\varphi : P \to P/K$ be the natural projection. We will prove by transfinite induction that

$$f(\varphi^{-1}(J^\alpha(P/K))) = \varphi^{-1}(J^\alpha(P/K)) \text{ for every ordinal } \alpha.$$

In fact, as $K \ll P$ and f is an automorphism, then

$$f(\varphi^{-1}(J(P/K))) = f(J(P)) = J(P) = \varphi^{-1}(J(P/K)).$$

We now assume that for each ordinal $\beta < \alpha$ the following equality holds

$$f(\varphi^{-1}(J^\beta(P/K))) = \varphi^{-1}(J^\beta(P/K)).$$

If α is a limit ordinal, then it is obvious that

$$f(\varphi^{-1}(J^\alpha(P/K))) = \varphi^{-1}(J^\alpha(P/K)).$$

We now suppose that α is not a limit ordinal. Call $\varphi' : P/K \to P/\varphi^{-1}(J^\alpha(P/K))$ the natural homomorphism. According to the inductive assumption

$$f(\varphi^{-1}(J^{\alpha-1}(P/K))) = \varphi^{-1}(J^{\alpha-1}(P/K)).$$

It follows that there is an isomorphism $f' : P/\varphi^{-1}(J^{\alpha-1}(P/K)) \to P/\varphi^{-1}(J^{\alpha-1}(P/K))$ such that the following diagram is commutative.

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\varphi"] & P \arrow[d, "\varphi"] \\ P/K \arrow[d, "\varphi'"] & P/K \arrow[d, "\varphi'"] \\ P/\varphi^{-1}(J^{\alpha-1}(P/K)) \arrow[r, "f'"] & P/\varphi^{-1}(J^{\alpha-1}(P/K)) \end{tikzcd}$$

In this case $f' \circ \varphi' \circ \varphi = \varphi' \circ \varphi \circ f$.

On the other hand, as P/K is a dual automorphism-extendable module, there exists a homomorphism $g : P/K \to P/K$ such that $\varphi' \circ g = f' \circ \varphi'$.

$$\begin{tikzcd} P/K \arrow[r, "g"] \arrow[d, "\varphi'"] & P/K \arrow[d, "\varphi'"] \\ P/\varphi^{-1}(J^{\alpha-1}(P/K)) \arrow[r, "f'"] & P/\varphi^{-1}(J^{\alpha-1}(P/K)) \end{tikzcd}.$$

Since P is projective, there is a homomorphism $g' : P \to P$ such that the following diagram commutative; that is $g \circ \varphi = \varphi \circ g'$.

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "\varphi"] & P \arrow[d, "\varphi"] \\ P/K \arrow[r, "g"] & P/K \end{tikzcd}$$

We obtain, from the above commutative diagrams, that

$$(f - g')(P) \leq \varphi^{-1}(J^{\alpha-1}(P/K)) \leq J(P) \ll P.$$

Furthermore, P is a projective module and hence $f - g' \in J(\mathrm{End}(P))$. We deduce that g' is an automorphism of P. Then g is an epimorphism. By our assumption, g is an automorphism of P/K. It follows that $g'(\mathrm{Ker}(\varphi)) = \mathrm{Ker}(\varphi)$. On the other hand, we can check that for every ordinal γ, then we have the equality $g(J^{\gamma}(P/K)) = J^{\gamma}(P/K)$. Consequently $g'(\varphi^{-1}(J^{\gamma}(P/K))) = \varphi^{-1}(J^{\gamma}(P/K))$.

$$\text{As } (f - g')(\varphi^{-1}(J^{\alpha-1}(P/K))) \leq (f - g')(J(P)) \leq J(\varphi^{-1}(J^{\alpha-1}(P/K)))$$
$$\text{and } \varphi(J(\varphi^{-1}(J^{\alpha-1}(P/K)))) \leq J(J^{\alpha-1}(P/K)) = J^{\alpha}(P/K), \text{ then}$$
$$(f - g')(\varphi^{-1}(J^{\alpha-1}(P/K))) \leq \varphi^{-1}(J^{\alpha}(P/K)).$$

It follows that

$$\begin{aligned} f(\varphi^{-1}(J^{\alpha}(P/K))) &\leq (f - g')(\varphi^{-1}(J^{\alpha}(P/K))) + g'(\varphi^{-1}(J^{\alpha}(P/K))) \\ &\leq \varphi^{-1}(J^{\alpha}(P/K)). \end{aligned}$$

We now suppose that $f(\varphi^{-1}(J^{\alpha}(P/K))) \neq \varphi^{-1}(J^{\alpha}(P/K))$. We have

$$f(\varphi^{-1}(J^{\alpha-1}(P/K))) = \varphi^{-1}(J^{\alpha-1}(P/K)) \text{ and obtain that there is an element}$$

$m \in \varphi^{-1}(J^{\alpha-1}(P/K))$ such that $m \notin \varphi^{-1}(J^{\alpha}(P/K))$ and $f(m) \in \varphi^{-1}(J^{\alpha}(P/K))$. As $f(m) = (f - g')(m) + g'(m)$ and $f(m), (f - g')(m) \in \varphi^{-1}(J^{\alpha}(P/K))$ then $g'(m) \in \varphi^{-1}(J^{\alpha}(P/K))$. On the other hand, g' is an automorphism and

$$g'(\varphi^{-1}(J^{\alpha}(P/K))) = \varphi^{-1}(J^{\alpha}(P/K)), \text{ consequently } m \in \varphi^{-1}(J^{\alpha}(P/K)).$$

This is a contradiction. It shows that $f(\varphi^{-1}(J^{\alpha}(P/K))) = \varphi^{-1}(J^{\alpha}(P/K))$. As R is a right perfect ring, then $K = \varphi^{-1}(J^{\alpha_0}(P/K))$ for some ordinal α_0. Consequently $f(K) = K$. We deduce that M is a dual automorphism-invariant module.

(2) $(a) \Rightarrow (b)$ by Corollary 2 in [17].

$(b) \Rightarrow (c)$ is clear.

$(c) \Rightarrow (a)$ Similar to the proof of $(b) \Rightarrow (a)$ of (1). □

Corollary 16. *Let R be a right artinian ring. The following conditions are equivalent for a finitely generated module M:*
(1) *M is a dual automorphism-extendable module.*
(2) *M is a dual automorphism-invariant module.*

A ring R is called right *invariant (duo)* if all right ideals of R are two-sided ideals. A ring R is called *abelian* if all idempotents of R are central.

Proposition 17. *Let R be an abelian right artinian ring. The following conditions are equivalent:*
(1) *Every cyclic right R-module is a dual automorphism-invariant module.*
(2) *Every cyclic right R-module is a dual automorphism-extendable module.*
(3) *Every cyclic right R-module is a quasi-projective module.*
(4) *R is a right invariant ring.*

Proof.
(1) $\Leftrightarrow$ (2) by Corollary 16.
(1) $\Rightarrow$ (4) It suffices to consider the case when the ring R is a local ring. Let A be a proper right ideal of R, and let $r \in R$. Since R is a local ring, either r is a unit or $1 - r$ is a unit. By [11, Theorem 2.6], then $rA \subseteq A$ or $(1 - r)A \subseteq A$. This implies that $rA \subseteq A$. We deduce that A is a left ideal of R. Thus R is a right invariant ring.
(4) $\Rightarrow$ (3) by [10, Theorem 10.13].
(3) $\Rightarrow$ (2) is clear. □

Theorem 18. *Assume that R is a right artinian ring. The following conditions are equivalent for a module M:*
(1) *M is a quasi-projective module.*
(2) *M is a dual automorphism-extendable module and has a decomposition $M = \oplus_I M_i$ with hollow modules M_i.*

Proof.
(1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (1) By [14, Proposition 4.32], we only need to show that each M_i is M-projective. Since M_i is hollow, it is cyclic. By Lemma 2, each M_i is a dual automorphism-extendable module, and so by Corollary 16, each M_i is a dual automorphism-invariant module. By [17, Theorem 30], each M_i is quasi-projective. Now, by Proposition 6 each M_i is M-projective as required. □

Remark 19. Note that we cannot drop the artinian assumption in the above theorem as $\mathbb{Z}_{p^\infty}$ is a hollow dual automorphism-extendable $\mathbb{Z}$-module but not quasi-projective.

A module M is called a *lifting* module if for every submodule N of M, there exists a decomposition $M = A \oplus B$ with $A \leq N$ and $N \cap B \ll B$. Clearly by Theorem 10 and [14, Theorem 4.15], if M is a lifting dual automorphism-extendable module, then, M is a direct sum of hollow modules. Now, the next two corollaries follow from Theorem 22.

Corollary 20. *Assume that R is a right artinian ring. The following conditions are equivalent:*
(1) *M is a lifting dual automorphism-extendable module.*
(2) *M is a quasi-projective module.*

Corollary 21. *Let R be an artinian serial ring. The following conditions are equivalent:*
(1) *M is a dual automorphism-extendable module.*
(2) *M is a quasi-projective module.*
(3) *M is a dual automorphism-invariant module.*

Theorem 22. *Assume that R is a right artinian ring and M, a finitely generated dual automorphism-extendable right R-module. If $M/J(M)$ is the direct sum of isomorphic simple modules, then M is a quasi-projective module.*

Proof. Suppose that R is a right artinian ring and $\{e_1, \dots, e_n\}$, a basic set of primitive idempotents of R and $M/J(M)$ is the direct sum of isomorphic simple modules. By our assumption, $M/J(M) = \bigoplus_{i=1}^{k} S_i$, where $S_i \simeq S_j$ for all i, j. Let $\pi : P \to M$ be a projective cover of M. It follows that $p \circ \pi : P \to M/J(M)$ is also a projective cover, where $p : M \to M/J(M)$ is the natural projection. Clearly, $P \simeq e_j R^k$ for some j.

Since M is dual automorphism-extendable, M is a dual automorphism-invariant module by Corollary 16. If P is a local module then M is a quasi-projective module by [17, Theorem 30]. Note that every projective module over right perfect rings is the direct sum of modules with local their endomorphism rings. Let e be an idempotent of $\mathrm{End}(P)$. By [6, Theorem 3.10] we have the decompositions $eP = \bigoplus_{i\in I_1} P_i$ and $(1-e)P = \bigoplus_{j\in I_2} P_j$, where $\mathrm{End}(P_i)$ are local rings and the sets I_1 and I_2 are disjoint finite sets. By Krull-Schmidt-Remak-Azumaya Theorem, there exists an isomorphism $f : P_i \to P_j$ for any $i, j \in I_1 \cup I_2$ with $i \neq j$.

Call $I = I_1 \cup I_2$ and $\pi_j : \bigoplus_{i\in I} P_i \to P_j$ the natural projections. Consider the following maps

$$\begin{array}{rcccl} \phi_1 & : & P & \longrightarrow & P \\ & & x_1 + x_2 + x_3 & \longmapsto & x_1 + (x_2 + f(x_1)) + x_3, \end{array}$$

$$\begin{array}{rcccl} \phi_2 & : & P & \longrightarrow & P \\ & & x_1 + x_2 + x_3 & \longmapsto & (x_1 + f^{-1}(x_2)) + x_2 + x_3, \end{array}$$

where $x_1 \in P_i, x_2 \in P_j, x_3 \in \bigoplus_{k\in I\setminus\{i,j\}} P_k$.

It is easy to see that ϕ_1 and ϕ_2 are automorphisms of P. It follows that $\phi_1(\mathrm{Ker}(\pi)) = \mathrm{Ker}(\pi)$ and $\phi_2(\mathrm{Ker}(\pi)) = \mathrm{Ker}(\pi)$ by [17, Theorem 27]. Let m be an arbitrary element of $\mathrm{Ker}(\pi)$. There exist elements $x_1 \in P_i, x_2 \in P_j, x_3 \in \bigoplus_{k\in I\setminus\{i,j\}} P_k$ such that $m = x_1 + x_2 + x_3$. Since

$$f(x_1) = \phi_1(m) - m \in \mathrm{Ker}(\pi),\ f^{-1}(x_2) = \phi_2(m) - m \in \mathrm{Ker}(\pi),$$

then

$$x_1 = \phi_2(f(x_1)) - f(x_1) \in \mathrm{Ker}(\pi),\ x_2 = \phi_1(f^{-1}(x_2)) - f^{-1}(x_2) \in \mathrm{Ker}(\pi).$$

We deduce that $x_3 \in \mathrm{Ker}(\pi)$. Thus

$$\mathrm{Ker}(\pi) = (P_i \cap \mathrm{Ker}(\pi)) \oplus (P_j \cap \mathrm{Ker}(\pi)) \oplus (P_k \cap \mathrm{Ker}(\pi))$$

with $P_k = \bigoplus_{l\in I\setminus\{i,j\}} P_l$. Then $\pi_i(\mathrm{Ker}(\pi)) \subset \mathrm{Ker}(\pi), \pi_j(\mathrm{Ker}(\pi)) \subset \mathrm{Ker}(\pi)$. It means that $\pi_i(\mathrm{Ker}(\pi)) \subset \mathrm{Ker}(\pi)$ for every $i \in I_1 \cup I_2$. Therefore, $e(\mathrm{Ker}(\pi)) \subset \mathrm{Ker}(\pi)$.

We now consider an arbitrary endomorphism f of P. By [3, Theorem 5.3] holds equality $f = e + g$, where $e \in \mathrm{End}(P)$ is an idempotent and g is an automorphism of P. Since M is a dual automorphism-invariant module, then $g(\mathrm{Ker}(\pi)) = \mathrm{Ker}(\pi)$. It shows that $\mathrm{Ker}(\pi)$ is a fully invariant submodule of P. Thus M is a quasi-projective module by [22, 18.2]. □

Corollary 23. *Let R be an abelian right artinian ring and M, a finitely generated module. The following conditions are equivalent:*

(1) *M is a dual automorphism-extendable module.*
(2) *M is a dual automorphism-invariant module.*
(3) *M is a quasi-projective module.*

Acknowledgments

The authors would like to thank the referees for the very helpful comments and suggestions.

Funding

Truong Cong Quynh has been partially founded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2017.22 and was supported by the Russian Government Program of Competitive Growth of Kazan Federal University.

References

[1] Amin, I., Ibrahim, Y., Yousif, M. F. (2014). D3-modules. *Commun. Algebra* 42:578–592.
[2] Anderson, F. W., Fuller, K. R. (1992). *Rings and Categories of Modules*. 2nd ed. GTM, Vol. 13. New York: Springer-Verlag.
[3] Camillo, V. P., Khurana, D., Lam, T. Y., Nicholson, W. K., Zhou, Y. (2006). Continuous modules are clean. *J. Algebra* 304:94–111.
[4] Clark, J., Lomp, C., Vanaja, N., Wisbauer, R. (2006). *Lifting Modules: Supplements and Projectivity in Module Theory*. Frontiers in Mathematics. Basel: Birkhauser Verlag.
[5] Er, N., Singh, S., Srivastava, A. K. (2013). Rings and modules which are stable under automorphisms of their injective hulls. *J. Algebra* 379:223–229.
[6] Facchini, A. (1998). *Module Theory. Endomorphism Rings and Direct Sum Decompositions in Some Classes of Modules*. Progress in Mathematics, Vol. 167. Basel: Birkhäuser Verlag.
[7] Fuchs, L. (1970). *Infinite Abelian Groups*, Vol. 1. New York: Academic Press.
[8] Guil Asensio, P. A., Keskin Tütüncü, D., Srivastava, A. K. (2015). Modules invariant under automorphisms of their covers and envelopes. *Israel J. Math.* 206:457–482.
[9] Guil Asensio, P. A., Srivastava, A. K. (2013). Automorphism-invariant modules satisfy the exchange property. *J. Algebra* 388:101–106.
[10] Jain, S. K., Srivastava, A. K., Tuganbaev, A. A. (2012). *Cyclic Modules and the Structures of Rings*. Oxford, UK: Oxford University Press.
[11] Koşan, M. T., Ha, N. T. T., Quynh, T. C. (2016). Rings for which every cyclic module is dual automorphism-invariant. *J. Algebra Appl.* 15:1650078 (11 pp.).
[12] Koşan, M. T., Quynh, T. C., Srivastava, A. K. (2016). Rings with each right ideal automorphism-invariant. *J. Pure Appl. Algebra* 220:1525–1537.
[13] Lee, T. K., Zhou, Y. (2013). Modules which are invariant under automorphisms of their injective hulls. *J. Algebra Appl.* 12:1250159 (9 pp.).
[14] Mohamed, S. H., Müller, B. J. (1990). *Continuous and Discrete Modules*. London Mathematical Society LN, Vol. 147. Cambridge, UK: Cambridge University Press.
[15] Nicholson, W. K., Yousif, M. F. (2003). *Quasi-Frobenius Rings*. Cambridge, UK: Cambridge University Press.
[16] Quynh, T. C., Koşan, M. T. (2015). On automorphism-invariant modules. *J. Algebra and Appl.* 14:1550074 (11 pp.).
[17] Singh, S., Srivastava, A. K. (2012). Dual automorphism-invariant modules. *J. Algebra* 371:262–275.
[18] Tuganbaev, A. A. (2013). Automorphisms of submodules and their extensions. *Discrete Math. Appl.* 23:115–124.
[19] Tuganbaev, A. A. (2013). Characteristic submodules of injective modules. *Discrete Math. Appl.* 23:203–209.
[20] Tuganbaev, A. A. (2015). Automorphism-extendable modules. *Discrete Math. Appl.* 25:305–309.
[21] Tuganbaev, A. A. (2017). Automorphism-invariant semi-Artinian modules. *J. Algebra Appl.* 16:1750029 (5 p.).
[22] Wisbauer, R. (1991). *Foundations of Module and Ring Theory*. Reading: Gordon and Breach.